CENTERS OF GENERIC HECKE ALGEBRAS

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ABSTRACT. Let W be a Weyl group and let W' be a parabolic subgroup of W. Define A as follows:

$$A = R \otimes_{Q[u]} \mathscr{A}(W)$$

where $\mathscr{A}(W)$ is the generic algebra of type A_n over $Q[u]$, u an indeterminate, associated with the group W, and R is a $Q[u]$-algebra, possibly of infinite rank, in which u is invertible. Similarly, we define A' associated with W'. Let M be an A-A bimodule, and let $b \in M$. Define the relative norm [14]

$$N_{W,W'}(b) = \sum_{t \in T} u^{-l(t)} a_{t^{-1}} b a_t$$

where T is the set of distinguished right coset representives for W' in W. We show that if $b \in Z_M(A') = \{m \in M | ma' = a'm \ \forall a' \in A'\}$, then $N_{W,W'}(b) \in Z_M(A)$. In addition, other properties of the relative norm are given and used to develop a theory of induced modules for generic Hecke algebras including a Markey decomposition. This section of the paper is previously unpublished work of P. Hoefsmit and L. L. Scott.

Let $\alpha = (k_1, k_2, \ldots, k_z)$ be a partition of n and let $S_\alpha = \prod_{i=1}^{z} S_{k_i}$ be a "left-justified" parabolic subgroup of S_n of shape α.

Define

$$b_\alpha = N_{S_n, S_\alpha}(\mathscr{N}_\alpha),$$

where

$$\mathscr{N}_\alpha = \prod_{i=1}^{z} N_{S_{k_i-1}, S_1}(a_{w_i})$$

with w_i a k_i-cycle of length $k_i - 1$ in S_{k_i}. Then the main result of this paper is

Theorem. *The set* $\{b_\alpha \mid \alpha \vdash n\}$ *is a basis for* $Z_{A(S_n)}(A(S_n))$ *over* $Q[u, u^{-1}]$.

Remark. The norms b_α in $Z_{A(S_n)}(A(S_n))$ are analogs of conjugacy class sums in the center of QS_n and, in fact, specialization of these norms at $u = 1$ gives the standard conjugacy class sum basis of the center of QS_n up to coefficients from Q.

1. PRELIMINARIES

This section contains certain key definitions and results, presented without proofs, of standard fare needed for later sections.

1.1. Parabolic subgroups of Weyl groups.

Let $W = W(\Delta)$ be a Weyl group and let $R = \{w_{\alpha_1}, w_{\alpha_2}, \ldots, w_{\alpha_n}\}$ with $\alpha_1, \alpha_2, \ldots, \alpha_n$ fundamental.

Definition (1.1.1). Let $J \leq R$. Then $W_J = \langle J \rangle$ is a parabolic subgroup of W.

Proposition (1.1.2). *Let* $I, J \leq R$. *Let* $W_I w W_J$ *be a double coset in* W. *Then there is a unique element* $\tilde{w}$ *of minimal length in* $W_I w W_J$ *such that, given any* $v \in W_I w W_J$, *there exist elements* $s \in W_I$ *and* $t \in W_J$ *so that* $v = s\tilde{w}t$, *with* $l(s\tilde{w}t) = l(s) + l(\tilde{w}) + l(t)$.

Definition (1.1.3). The element $\tilde{w}$ in Proposition (1.1.2) is called a distinguished double coset representative.

Note. If $J = \varnothing$, then $\tilde{w}$ is called a distinguished right coset representative, adopting the convention $W_\varnothing = \{1\}$.

Proposition (1.1.4). *Let* I, $J < R$ *and let* $\tilde{w}$ *be distinguished in* $W_I w W_J$. *Then* $(W_I)^{\tilde{w}} \cap W_J$ *is a parabolic subgroup of* W.

Theorem (1.1.5) (Exchange condition). *If* $w_{\alpha_1} w_{\alpha_2} \cdots w_{\alpha_k}$ *is reduced and*

$$w_{\alpha_0} w_{\alpha_1} w_{\alpha_2} \cdots w_{\alpha_k}$$

is not reduced $(\alpha_0 \in \Pi)$, *then for some* $i \leq k$,

$$w_{\alpha_1} w_{\alpha_2} \cdots w_{\alpha_i} = w_{\alpha_0} w_{\alpha_1} w_{\alpha_2} \cdots w_{\alpha_{i-1}}.$$

Remark. Given $w_{\alpha_1} w_{\alpha_2} \cdots w_{\alpha_k}$ reduced with $w_{\alpha_0} w_{\alpha_1} w_{\alpha_2} \cdots w_{\alpha_k}$ not reduced, the exchange condition implies that

$$(w_{\alpha_0} w_{\alpha_1} \cdots w_{\alpha_{i-1}}) w_{\alpha_i} w_{\alpha_{i+1}} \cdots w_{\alpha_k} = (w_{\alpha_1} \cdots w_{\alpha_i}) w_{\alpha_i} w_{\alpha_{i+1}} \cdots w_{\alpha_k}$$

$$= w_{\alpha_1} \cdots w_{\alpha_{i-1}} w_{\alpha_{i+1}} \cdots w_{\alpha_k}.$$

1.2. The generic algebra and specialized algebra.

Let (W, R) be a finite Coxeter system with $R = \{w_1, w_2, \ldots, w_n\}$. Let $D = \mathbf{Q}[u_1, u_2, \ldots, u_n]$ be the polynomial ring over $\mathbf{Q}$ in indeterminates $\{u_i\}$ corresponding to the distinguished generators in R and satisfying the condition that $u_i = u_j$ whenever w_i and w_j are conjugate in W.

Definition (1.2.1). The generic algebra (or generic Hecke algebra) $\mathscr{A} = \mathscr{A}(W)$ is the algebra over D, with basis $\{a_w \mid w \in W\}$ over D, identity element a_1, and multiplication defined by the conditions

$$a_{w_i} a_w = \begin{cases} a_{w_i w}, & l(w_i w) \geq l(w), \\ u_i a_{w_i w} + (u_i - 1) a_w, & l(w_i w) < l(w). \end{cases}$$

$\mathscr{A}$ has a presentation with generators $\{a_{w_i} \mid w_i \in R\}$ and quadratic relations

$$a_{w_i}^2 = u_i a_1 + (u_i - 1) a_{w_i}, \qquad w \in R,$$

and homogeneous relations

$$a_{w_i} a_{w_j} a_{w_i} \cdots = a_{w_j} a_{w_i} a_{w_j} \cdots \qquad (i \neq j)$$

with n_{ij} factors, where n_{ij} is the order of the product $w_i w_j$ in W. Thus we may define the following homomorphism:

$$\text{ind} : \mathscr{A} \to D$$

$$a_{w_i} \mapsto u_i.$$

We then may define the Poincaré polynomial of W, and generic degree corresponding to an irreducible character χ of $\overline{K} \otimes_R \mathscr{A}(W)$ where $\overline{K}$ is an algebraic closure of K.

Definition (1.2.2). Let $\mathscr{A} = \mathscr{A}(W)$ be a generic algebra, and χ an irreducible character of $\overline{K} \otimes_R \mathscr{A}$. The generic degree corresponding to χ is the element of $\overline{K}$ given by

$$d_\chi = \frac{(\deg \chi) \cdot d_W}{\sum_{w \in W} (\text{ind}(a_w))^{-1} \chi(a_w) \chi(a_{w^{-1}})},$$

where d_W is the Poincaré polynomial of W and is defined by

$$d_W = \sum_{w \in W} \text{ind}(a_w).$$

Definition (1.2.3). Let L be any field of characteristic zero and $\phi : D \to L$ a homomorphism. Consider L as an L-D bimodule by setting

$$l\lambda d = l\lambda\phi(d), \qquad l, \lambda \in L, \ d \in D.$$

Then $\mathscr{A}_{f,L} = L \otimes_D \mathscr{A}$ is an associative algebra over L called a specialized algebra.

$\mathscr{A}_{f,L}$ has a basis $\{a_{w\phi} = 1 \otimes a_w \mid w \in W\}$, generators $\{a_{w_i\phi} \mid w_i \in R\}$, and defining relations obtained by applying ϕ to the relations for $\mathscr{A}$ in Definition (1.2.1).

Proposition (1.2.4). *Let* $f : D \to \mathbf{Q}$ *be defined by* $f(u_i) = q_i \ \forall i$. *Then* $\mathscr{A}_{f,\mathbf{Q}} \simeq \mathscr{H}(G, B)$.

Let $f : D \to \mathbf{Q}$ *be defined by* $f(u_i) = 1 \ \forall i$. *Then* $\mathscr{A}_{f,\mathbf{Q}} \simeq \mathbf{Q}W$.

Theorem (1.2.5) (Tits). *Let* $G = \text{SL}(n, q)$, $B = $ *subgroup of* G *consisting of upper triangular matrices and* $W = S_n$. *Then* $\mathscr{H}(G, B) \simeq \mathbf{C}W$.

Remark. In [11], Curtis and Scott give a (modular-theoretic) proof of the fact that $\mathscr{H}(G, B) \simeq \mathbf{Q}W$, with G, B and W as above, along the lines of Feit [12] which motivated the work of Hoefsmit and Scott [14] presented in the next section.

2. GENERAL RESULTS

The material in this section is due mostly to Peter Hoefsmit and Leonard Scott, taken from unpublished work [14] done in 1976. We would like at this time to point out the results of this section which the reader should focus on in order to understand the main results of this paper, given in §3. They are Lemma (2.12), Proposition (2.13), and Theorem (2.30).

Definition (2.1). Let u be an indeterminate. Define

$$A = R \otimes_{\mathbf{Q}[u]} \mathscr{A},$$

where R is a $\mathbf{Q}[u]$-algebra, possibly of infinite rank, in which u is invertible and where $\mathscr{A} = \mathscr{A}(W)$ is the generic algebra of a Weyl group W over $\mathbf{Q}[u]$. Similarly, define

$$A' = R \otimes_{\mathbf{Q}[u]} \mathscr{A}',$$

where $\mathscr{A}' = \mathscr{A}'(W')$ is the generic subalgebra of $\mathscr{A}$ associated with a parabolic subgroup W' of W.

Definition (2.2). Let M be an A'-A' bimodule. Define

$$Z_M(A') = \{m \in M \mid a'm = ma' \text{ for all } a' \in A'\}.$$

Note. We will write $Z(A)$ for $Z_A(A)$.

Remark. If W_1 and W_2 are parabolic subgroups of W, then A_1 and A_2 will denote the corresponding subalgebras of A unless otherwise stated. Also, given an arbitrary parabolic subgroup P of W, we will sometimes denote the corresponding subalgebra of A as $A(P)$.

Definition (2.3). Let M be an A-A bimodule and let $b \in M$. Let R be the set of distinguished right coset representatives for W' in W. Define the *relative norm*

$$N_{W,W'}(b) = \sum_{r \in R} u^{-l(r)} a_{r^{-1}} b a_r.$$

Note. If W' is the identity subgroup, we will write $N_{W,1}(b)$.

Lemma (2.4). *Let M be an A-A bimodule, and let $b \in M$. Then $N_{W,1}(b) \in Z_M(A)$.*

Proof. Consider the A-A bimodule $M \oplus A$. For each $m \in M$ we can define $\phi_m \in \mathrm{Hom}_{\mathbf{Q}(u)}(M \oplus A, M \oplus A)$ by

$$\phi_m(m' + a) = ma \quad \text{for any } m' \in M.$$

Thus $M \leq \mathrm{Hom}_{\mathbf{Q}(u)}(M \oplus A, M \oplus A)$, and since $N_{W,1}(b) \in M$, we have $N_{W,1}(b) \in \mathrm{Hom}_{\mathbf{Q}(u)}(M \oplus A, M \oplus A)$. Now $\{a_w \mid w \in W\}$ and $\{u^{-l(w)} a_{w^{-1}} \mid w \in W\}$ are dual bases for the symmetric algebra $\mathbf{Q}(u)A$ with respect to the non-degenerate associative *coefficient at the identity* form. Hence $\mathbf{Q}(u)A$ is Frobenius and therefore $N_{W,1}(b) \in \mathrm{Hom}_{\mathbf{Q}(u)A}(M \oplus A, M \oplus A)$ [8]. In particular, $aN_{W,1}(b) = N_{W,1}(b)a$ for all $a \in A$. Hence, $N_{W,1}(b) \in Z_M(A)$. $\square$

Lemma (2.5). *Let a_1 be the identity in A. Then $N_{W,1}(a_1)$ is invertible in $\mathbf{Q}(u)A$. (Note. By [13], $\mathbf{Q}(u)$ is a splitting field for A when $\mathscr{A}(w)$ is type A_n; for other situations extend to a splitting field.)*

Proof. Write

$$N_{W,1}(a_1) = \sum_{w \in W} u^{-l(t)} a_{w^{-1}} a_w = \sum_\chi c_\chi e_\chi,$$

where the e_χ are the central primitive idempotents associated with the irreducible characters for A over $\mathbf{Q}(u)$. We will determine c_χ explicitly. Applying a particular character χ to the norm, we get

$$(2.6) \qquad \sum_{w \in W} u^{-l(w)} \chi(a_{w^{-1}} a_w) = c_\chi \chi(a_1).$$

Let $\{X_{ij} \mid 1 \le i, j \le \chi(a_1)\}$ be a set of matrix coordinate functions associated with χ and let $\{f_{ij} \mid 1 \le i, j \le \chi(a_1)\}$ be the set of corresponding matrix units. Then

$$(2.7) \qquad \sum_{w \in W} u^{-l(w)} \chi(a_{w^{-1}} a_w) = \sum_{w \in W} \sum_{i,j} u^{-l(w)} X_{ji}(a_{w^{-1}}) X_{ij}(a_w)$$

$$= \sum_{w \in W} \sum_{i,j} u^{-l(w)} \chi(f_{ij} a_{w^{-1}}) X_{ij}(a_w).$$

Now write $f_{ij} = \sum_{w \in W} c_w a_w$. Consider the function $\rho = \sum_\chi d_\chi \chi$, where d_χ is the generic degree of χ. Observe that

$$\rho(a_w) = \begin{cases} d_W & \text{(where } d_W \text{ is the Poincaré polynomial for } W) \\ & \text{if } w = \text{ the identity,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tau \in W$. Then

$$\rho(f_{ij} a_\tau) = \rho\left(\left(\sum_{w \in W} c_w a_w\right) a_\tau\right) = \begin{cases} c_w u^{l(w)} d_W & \text{if } \tau = w^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Also,

$$\rho(f_{ij} a_{w^{-1}}) = d_\chi \chi(f_{ij} a_{w^{-1}}).$$

Thus

$$c_w = d_W^{-1} d_\chi u^{-l(w)} \chi(f_{ij} a_{w^{-1}}),$$

and hence

$$(2.8) \qquad f_{ij} = \sum_{w \in W} d_W^{-1} d_\chi u^{-l(w)} \chi(f_{ij} a_{w^{-1}}) a_w.$$

Multiplying both sides of (2.8) by f_{ji} and applying χ gives

$$(2.9) \qquad \chi(f_{ji} f_{ij}) = \sum_{w \in W} d_W^{-1} d_\chi u^{-l(w)} \chi(f_{ij} a_{w^{-1}}) \chi(f_{ji} a_w).$$

Summing both sides of (2.9) over i and j, we have

$$(2.10) \qquad \chi(a_1)^2 = \sum_{i,j} \sum_{w \in W} d_W^{-1} d_\chi u^{-l(w)} \chi(f_{ij} a_{w^{-1}}) \chi(f_{ji} a_w),$$

since $f_{ji} f_{ij} = f_{jj}$. Combining (2.10) with (2.7) yields

$$(2.11) \qquad \chi(a_1)^2 = d_W^{-1} d_\chi \sum_{w \in W} u^{-l(w)} \chi(a_{w^{-1}} a_w).$$

Finally, from (2.11) and (2.6) we deduce that $c_\chi = d_W^{-1} d_\chi \chi(a_1)$, which is invertible in $\mathbf{Q}(u)$ and the lemma follows. $\square$

We see that our norm has a nice transitivity property with respect to parabolic subgroups.

Lemma (2.12). *Let M be an A-A bimodule and let $b \in M$. Let W' and W'' be parabolic subgroups of W such that $W' \leq W'' \leq W$. Then*

$$N_{W,W'}(b) = N_{W,W''}(N_{W'',W'}(b)).$$

Proof. Let R be the set of distinguished right coset representatives for W'' in W, let T be the set of distinguished right coset representatives for W' in W'' and let S be the set of distinguished right coset representatives for W' in W. Write

$$W = \bigcup_{r \in R} W'' r = \bigcup_{r \in R} \left(\bigcup_{t \in T} W' t \right) r.$$

From this decomposition we see that $tr \in S$. That is, since r is distinguished in $W'' r$ and t is distinguished in $W' t$, it follows that tr is distinguished in $W' tr$. Also, from Proposition (1.1.2), we have $l(tr) = l(t) + l(r)$ since $t \in W''$. Thus

$$
\begin{aligned}
N_{W,W''}(N_{W'',W'}(b)) &= \sum_{r \in R} u^{-l(r)} a_{r^{-1}} \left[\sum_{t \in T} u^{-l(t)} a_{t^{-1}} b a_t \right] a_r \\
&= \sum_{r \in R} \sum_{t \in T} u^{-l(tr)} a_{r^{-1} t^{-1}} b a_{tr} \\
&= \sum_{s \in S} u^{-l(a)} a_{s^{-1}} b a_s = N_{W,W'}(b). \quad \square
\end{aligned}
$$

Proposition (2.13). *Let M be an A-A bimodule and let $b \in Z_M(A')$. Then*

(1) $N_{W,W'}(b) \in Z_M(A)$;

(2) *If $Z_M(A)$ is regarded as a $Z(A)$-module, then $N_{W,W'}(Z_M(A'))$ is a $Z(A)$-submodule of $Z_M(A)$.*

Proof. (1) Clearly $N_{W,W'}(b) \in M$. Now

$$
\begin{aligned}
N_{W,W'}(b) &= N_{W,W'} \left(\frac{N_{W',1}(a_1)}{N_{W',1}(a_1)} b \right) \quad \text{by Lemma (2.5)} \\
&= N_{W,W'} \left(N_{W',1} \left(\frac{a_1}{N_{W',1}(a_1)} \right) b \right) \quad \text{by Lemma (2.4)} \\
&= N_{W,W'} \left(N_{W',1} \left(\frac{b}{N_{W',1}(a_1)} \right) \right) \quad \text{since } b \in Z_M(A') \\
&= N_{W,1} \left(\frac{b}{N_{W',1}(a_1)} \right) \quad \text{by Lemma (2.12)}.
\end{aligned}
$$

But $N_{W,1}(b/N_{W',1}(a_1)) \in Z_{\mathbf{Q}(u)A \cdot M}(A)$ by Lemma (2.4), and since

$$Z_{\mathbf{Q}(u)A \cdot M}(A) \leq Z_{\mathbf{Q}(u)M}(A),$$

we have

$$N_{W,W'}(b) \in M \cap Z_{\mathbf{Q}(u)M}(A) = Z_M(A),$$

which proves (1).

(2) For $z \in Z(A)$, we have

$$N_{W,W'}(b)z = N_{W,W'}(bz),$$

which proves (2). □

Lemma (2.14). *Let a_w be any basis element of A. Then a_w is invertible in A.*

Proof. Since w can be expressed as a product of fundamental reflections, we may assume without loss of generality that w itself is a fundamental reflection. From the relation $a_w^2 = ua_1 + (u-1)a_w$, we easily derive

$$a_w^{-1} = u^{-1}(a_w - (u-1)a_1) \in A. \quad \Box$$

Lemma (2.15). *Let W_1 and W_2 be two parabolic subgroups of W such that $W_2 = W_1^w$ for some $w \in W$. Let N be an A-A_1 bimodule. Consider the A-A_2 bimodule $N \otimes_{A_1} a_d$ where d is distinguished in $W_1 w W_2 = W_1 w = w W_2$. Then*

$$N \otimes_{A_1} A \simeq (N \otimes_{A_1} a_d) \otimes_{A_2} A.$$

Proof. First note that $W_2 = W_1^d$ so that $W_1 d W_2 = W_1 d = W_2$ and $N \otimes_{A_1} a_d$ really is an A-A_2 bimodule. Now $(N \otimes_{A_1} a_d) \otimes_{A_2} A \simeq N \otimes_{A_1} (A_1 a_d \otimes_{A_2} A)$, so it will be enough to show that $A_1 a_d \otimes_{A_2} A \simeq A$. Define maps

$$\phi: A_1 a_d \otimes_{A_2} A \to A \qquad \psi: A \to A_1 a_d \otimes_{A_2} A$$
$$a' a_d \otimes b \mapsto a' a_d b, \qquad a \mapsto a_d \otimes a_d^{-1} a.$$

Since $a_d^{-1} \in A$ by Lemma (2.14), ψ makes sense. Let $\hat{a} \in A_1$ and $\tilde{a} \in A$. Then $\hat{a} a_d = a_d \bar{a}$ for some $\bar{a} \in A_2$. Thus

$$\phi(\hat{a} \cdot a' a_d \otimes b) = \hat{a} a' a_d b = \hat{a} \cdot \phi(a' a_d \otimes b),$$

and

$$\phi(a' a_d \otimes b \cdot \tilde{a}) = a' a_d b \tilde{a} = \phi(a' a_d \otimes b) \cdot \tilde{a},$$

which shows that ϕ is an A_1-A bimodule map. Also

$$\psi(\hat{a} \cdot a) = a_d \otimes a_d^{-1} \hat{a} a = a_d \otimes \bar{a} a_d^{-1} a$$
$$= a_d \bar{a} \otimes a_d^{-1} a = \hat{a} a_d \otimes a_d^{-1} a = \hat{a} \psi(a)$$

and

$$\psi(a \cdot \tilde{a}) = a_d \otimes a_d^{-1} a \tilde{a} = \psi(a) \cdot \tilde{a}$$

so that ψ is an A_1-A bimodule map. Finally,

$$(\phi \circ \psi)(a) = \phi(a_d \otimes a_d^{-1} a) = a$$

and

$$
\begin{aligned}
(\psi \circ \phi)(\hat{a} a_d \otimes b) &= \psi(\hat{a} a_d b) = a_d \otimes a_d^{-1} \hat{a} a_d b \\
&= a_d a_d^{-1} \hat{a} a_d \otimes b \quad \text{since } a_d^{-1} \hat{a} a_d \in A_2, \\
&= \hat{a} a_d \otimes b,
\end{aligned}
$$

and the lemma follows.   □

Lemma (2.16). *Let R be the set of distinguished right coset representatives for W' in W. Then*

$$A = \bigoplus_{r \in R} A' a_r.$$

Proof. This follows immediately from the decomposition $W = \dot{\bigcup}_{r \in R} W'_r$ and Proposition (1.1.2).   □

Lemma (2.17). *Let W_1 and W_2 be parabolic subgroups of W and let $W_1 d W_2$ be a double coset in W with d distinguished. Then for any $w_1 \in W_1$, we have $l(w_1 d) = l(w_1) + l(d)$. Similarly, for any $w_2 \in W_2$, we have $l(d w_2) = l(d) + l(w_2)$.*

Proof. We give the proof for W_2. Suppose there is $v \in W_2$ such that $l(dv) < l(d) + l(v)$. Write $v = v_1 v_2 \cdots v_z$ as a reduced product of fundamental reflections v_i. Then there is a smallest subscript, say j, such that $l(d v_1 v_2 \cdots v_{j-1}) = l(d) + l(v_1 \cdots v_{j-1})$, but $l(d v_1 v_2 \cdots v_j) = l(d) + l(v_1 \cdots v_{j-1}) - 1$. Let $\hat{v} = v_1 v_2 \cdots v_{j-1}$. By Theorem (1.1.5), we have

$$
d\hat{v}v_j = \begin{cases}
d\hat{v}', & \text{where } l(\hat{v}') = l(\hat{v}) - 1 \\
& \text{if the exchange takes place in } \hat{v}, \\
d'\hat{v}, & \text{where } l(d') = l(d) - 1 \\
& \text{if the exchange takes place in } d.
\end{cases}
$$

The first case cannot happen since $l(\hat{v}v_j) = l(\hat{v}) + 1$. If $d\hat{v}v_j = d'\hat{v}$ with $l(d') = l(d) - 1$, then $d' = d\hat{v}v_j\hat{v}^{-1} \in W_1 d W_2$, which contradicts the fact that d is distinguished. This proves the lemma.   □

Remark. Let $d \in W_1 d W_2$ be distinguished. If $d^{-1} \in W_2 d^{-1} W_1$ is not distinguished, then for some $w_2 \in W_2$, $w_1 \in W_1$ we have

$$l(d) = l(d^{-1}) > l(w_2 d^{-1} w_1) = l((w_2 d^{-1} w_1)^{-1}) = l(w_1^{-1} d w_2^{-1}),$$

contradicting the fact that d is distinguished.

Definition (2.18). Let T be a right A-module. Then T is induced from a right A'-module T' if $T \simeq T' \otimes_{A'} A$.

Lemma (2.19). *Let T be a right A-module. Let R be the set of distinguished right coset representatives for W' in W. Then T is induced from the right A'-module T' if and only if $T = \bigoplus_{r \in R} Sa_r$ for some right A'-submodule S of T with $S \simeq T'$.*

Proof. Assume first that $T \simeq T' \otimes_{A'} A$ and let ϕ be an isomorphism from $T' \otimes_{A'} A$ to T. By Lemma (2.16) we have $T' \otimes_{A'} A = \bigoplus_{r \in R}(T' \otimes_{A'} a_1)a_r$ so take S to be $\phi(T' \otimes_{A'} a_1)$.

Now assume that $T = \bigoplus_{r \in R} Sa_r$. We have

$$\bigoplus_{r \in R} Sa_r \simeq \bigoplus_{r \in R}(S \otimes_{A'} a_r) \simeq S \otimes_{A'} \left(\bigoplus_{r \in R} Aa_r \right)$$
$$= S \otimes_{A'} A \quad \text{by Lemma (2.16)}$$

and the lemma follows. $\square$

Corollary (2.20). *Let W_1 and W_2 be parabolic subgroups of W and let $W_1 d W_2$ be a double coset in W with d distinguished. Then the A_1-A_2 bimodule $A_1 a_d A_2$ is isomorphic to $A_1 a_d \otimes_{A(W_1^d \cap W_2)} A_2$.*

Proof. Let T be the set of distinguished right coset representatives for $W' = W_1^d \cap W_2$ in W_2. Then

$$A_1 a_d A_2 = A_1 a_d \left(\bigoplus_{t \in T} A' a_t \right) \quad \text{by Lemma (2.16)}$$
$$= \bigoplus_{t \in T} A_1 a_d A' a_t.$$

Let $w' \in W'$. Then $w' = d^{-1} w_1 d$ for some $w_1 \in W_1$. By Lemma (2.17), $l(w_1 d) = l(w_1) + l(d)$. Also, since $w' \in W_2$, $l(dw') = l(d) + l(w')$ by Lemma (2.17). Thus there is a corresponding relation in A. In particular, given an element in $A_1 a_d A'$, say $\alpha a_d \alpha'$, with $\alpha \in A_1$, $\alpha' \in A'$, we may write it as $\alpha \hat{\alpha} a_d$ for some $\hat{\alpha} \in A_1$. Therefore $A_1 a_d A' \leq A_1 a_d$. Clearly $A_1 a_d \leq A_1 a_d A'$ and so $A_1 a_d A' = A_1 a_d$. Hence $\bigoplus_{t \in T} A_1 a_d A' a_t = \bigoplus_{t \in T} A_1 a_d a_t$. Now apply Lemma (2.19) with $S = A_1 a_d$ to get the result. $\square$

Lemma (2.21). *Let W_1 and W_2 be parabolic subgroups of W. Let D be the set of distinguished W_1-W_2 double coset representatives in W. Let R be the set of distinguished right coset representatives for W_1 in W. For $d \in D$, let T_d be the set of distinguished right coset representatives for $W_1^d \cap W_2$ in W_2. Then for any $t \in T_d$, we have*

    (1) $dt \in R$,
    (2) $l(dt) = l(d) + l(t)$.

Proof. (1) Suppose $dt \notin R$. Then $l(w_1 dt) < l(dt)$ for some $w_1 \in W_1$, so choose among all $w_1 \in W_1$ with $l(w_1 dt) < l(dt)$ an element $\hat{w}_1$ of minimal length. Write $\hat{w}_1 = \tilde{w}_1 v$ with $\tilde{w}_1 v$ reduced and $\tilde{w}_1$ a fundamental reflection. Thus

$$l(\hat{w}_1 dt) = l(\tilde{w}_1 vdt) < l(dt) \leq l(vdt).$$

Since $\tilde{w}_1$ is fundamental, Theorem (1.1.5) implies that one of three cases occurs:

$$\tilde{w}_1 vdt = \begin{cases} v'dt, & \text{where } l(v') = l(v) - 1 \\ & \text{if the exchange takes place in } v, \\ vd't, & \text{where } l(d') = l(d) - 1 \\ & \text{if the exchange takes place in } d, \\ vdt', & \text{where } l(t') = l(t) - 1 \\ & \text{if the exchange takes place in } t. \end{cases}$$

The first case cannot happen since $\tilde{w}_1 v$ is reduced. If $\tilde{w}_1 vdt = vd't$, then $d' = v^{-1}\tilde{w}_1 vd \in W_1 dW_2$ which contradicts the fact that d is distinguished. If $\tilde{w}_1 vdt = vdt'$, then $t' = (d^{-1}v^{-1}\tilde{w}_1 vd)t \in (W_1^d \cap W_2)t$, contradicting the fact that t is distinguished. Therefore $dt \in R$.

(2) Since $t \in W_2$, this follows immediately from Lemma (2.17). $\square$

Lemma (2.22). *Let M be an A-A bimodule and $N \leq M$ an A-A_1 bimodule. Let W_2 be any parabolic subgroup of W. Suppose $M \simeq N \otimes_{A_1} A$. Then M may be decomposed into chunks that are A_2 right-stable:* $M \simeq \sum_{d \in W_1 \backslash W / W_2, \text{ distinguished}} Na_d A_2$.

(*Note.* This is also a decomposition into A_2 left-stable pieces since N is a left A-module.)

Proof. Let D be the set of distinguished W_1-W_2 double coset representatives in W and let R be the set of distinguished right coset representatives for W_1 in W. Decompose W into W_1-W_2 double cosets:

$$W = \bigcup_{d \in D}^{\cdot} W_1 dW_2.$$

then

$$R = \bigcup_{d \in D}^{\cdot} (R \cap W_1 dW_2).$$

For a fixed $d \in D$, let T_d be the set of distinguished right coset representatives for $W_1^d \cap W_2$ in W_2. Then $dT_d \leq R$ by Lemma (2.21). Also, $dT_d \leq W_1 dW_2$ since $T_d \leq W_2$. Thus $dT_d \leq R \cap W_1 dW_2$. Now let $r \in R \cap W_1 dW_2$. Since $r \in W_1 dW_2$, we may write it as $w_1 dw_2$ where $w_1 \in W_1$, $w_2 \in W_2$, and $l(w_1 dw_2) = l(w_1) + l(d) + l(w_2)$ by Proposition (1.3.2). Since $r \in R$, r is distinguished in $W_1 r = W_1 w_1 dw_2$, and so $l(w_1^{-1} w_1 dw_2) = l(w_1^{-1}) + l(w_1 dw_2)$

by Proposition (1.1.2). But $w_1^{-1} w_1 d w_2 = d w_2$ and $l(d w_2) = l(d) + l(w_2)$ by Lemma (2.17). Combining these facts we have

$$l(d) + l(w_2) = l(d w_2) = l(w_1^{-1} w_1 d w_2)$$
$$= l(w_1^{-1}) + l(w_1 d w_2) = l(w_1^{-1}) + l(w_1) + l(d) + l(w_2),$$

which implies that $l(w_1) = 0$. Hence w_1 is the identity and $r = d w_2$.

Suppose $w_2 \notin T_d$. Then $l(d^{-1} w_1 d w_2) < l(w_2)$ for some $d^{-1} w_1 d \in W_1^d \cap W_2$. Since $d^{-1} w_1 d w_2 \in W_2$, we have by Lemma (2.17) that $l(d(d^{-1} w_1 d w_2)) = l(d) + l(d^{-1} w_1 d w_2)$. Thus

$$l(w_1 d w_2) = l(d(d^{-1} w_1 d w_2)) < l(d) + l(w_2) = l(d w_2).$$

But $d(d^{-1} w_1 d w_2) = w_1 d w_2 \in W_1 d w_2$ and $d w_2 \in R$. Hence $l(w_1 d w_2) > l(d w_2)$, which gives a contradiction. Therefore $r \in d T_d$ so that

$$d T_d = R \cap W_1 d W_2.$$

Thus

$$\sum_{r \in R \cap W_1 d W_2} N a_r = \sum_{t \in T_d} (N a_d) a_t.$$

Since $\bigoplus_{t \in T_d} (A_1 a_d) a_t = A_1 a_d A_2$ (as in the proof of Corollary (2.20)), we have

$$\sum_{t \in T_d} (N a_d) a_t = N a_d A_2$$

which is A_2 right-stable. So

$$N \otimes_{A_1} A = \sum_{r \in R} N a_r = \sum_{d \in D} \sum_{t \in T_d} (N a_d) a_t = \sum_{d \in D} N a_d A_2. \quad \square$$

Theorem (2.23). *Let M be an A-A bimodule and N an A-A' bisubmodule of M such that $M \simeq N \otimes_{A'} A$. Then $Z_M(A) = N_{W,W'}(Z_N(A'))$.*

Proof. By Proposition (2.13) we have that $N_{W,W'}(Z_M(A')) \le Z_M(A)$. Since $N \le M$, it follows that $Z_N(A') \le Z_M(A')$, and hence $N_{W,W'}(Z_N(A')) \le N_{W,W'}(Z_M(A')) \le Z_M(A)$. So we must show that $Z_M(A) \le N_{W,W'}(Z_N(A'))$. First we consider the case $W = \langle w \rangle$ where w is a single fundamental reflection and $W' = \{1\}$. Let $m \in Z_M(A)$. Since $M \simeq N \otimes_{A'} A \simeq N \oplus N a_w$, write m as $m = n_0 + n_1 a_w$ with $n_0, n_1 \in N$. Since $a_w m = m a_w$, we get

$$a_w(n_0 a_1 + n_1 a_w) = (a_w n_0) a_1 + (a_w n_1) a_w$$
$$= (n_0 a_1 + n_1 a_w) a_w = n_0 a_w + n_1 a_w^2$$
$$= n_0 a_w + n_1 (u a_1 + (u - 1) a_w)$$
$$= (u n_1) a_1 + ((u - 1) n_1 + n_0) a_w.$$

Equating terms in N gives $a_w n_0 = u n_1$. Then $a_w n_0 a_w = u n_1 a_w$ and so $n_1 a_w = u^{-1} a_w n_0 a_w$. Thus

$$m = n_0 + u^{-1} a_w n_0 a_w = N_{W',W''}(n_0) \in N_{W',W''}(Z_N(A')).$$

Now for the general situation write $N \otimes_{A'} A \simeq \bigoplus_{r \in R} N a_r$ where R is the set of distinguished right coset representatives for W' in W. Let $m \in Z_M(A)$. Write $m = n_1 + \sum_{r \in R, r \neq 1} n_r a_r$, $n_r \in N$. We will show that $m \in N_{W',W''}(Z_N(A'))$ by showing that $n_1 \in Z_N(A')$ and $m = N_{W',W''}(n_1)$.

Let $w \in W'$ be a fundamental reflection. Then $a_w m = m a_w$ which gives

$$a_w \left(n_1 + \sum_{\substack{r \in R \\ r \neq 1}} n_r a_r \right) = a_w n_1 + \sum_{\substack{r \in R \\ r \neq 1}} a_w n_r a_r$$

$$= \left(n_1 + \sum_{\substack{r \in R \\ r \neq 1}} n_r a_r \right) a_w = n_1 a_w + \sum_{\substack{r \in R \\ r \neq 1}} n_r a_r a_w.$$

Equating terms in N we get that $a_w n_1 = n_1 a_w$. Thus $n_1 \in Z_N(A')$.

Now let $r \in R$ be minimum in the sense that for all $\tilde{r} \in R$ with $l(\tilde{r}) < l(r)$ we know that

$$n_{\tilde{r}} a_{\tilde{r}} = u^{-l(\tilde{r})} a_{\tilde{r}-1} n_1 a_{\tilde{r}}.$$

Let $w_1 w_2 \cdots w_k$ be a reduced expression for r with the w_i fundamental reflections. Suppose that $w_1 w_2 \cdots w_{k-1} \in W'\hat{r}$, $\hat{r} \in R$. Then $w_1 w_2 \cdots w_{k-1} = w'\hat{r}$ for some $w' \in W'$ and $l(w'\hat{r}) = l(w') + l(\hat{r})$. But then $r = w_1 w_2 \cdots w_{k-1} w_k = w'\hat{r}w_k$ with $l(w'\hat{r}w_k) = l(w') + l(\hat{r}) + l(w_k)$ and $w'^{-1}(w'\hat{r}w_k)$, whose length is $l(\hat{r}) + l(w_k)$, is an element in $W'r$. Hence, since r is distinguished in Wr, it must be that $l(w') = 0$ and so $w' = 1$. That is, $w_1 w_2 \cdots w_{k-1} = \hat{r}$. Let $W_k = \langle w_k \rangle$. Note that $\hat{r}$ is distinguished in $W'\hat{r}W_k$ in W. Also $(W')^{\hat{r}} \cap W_k$ is trivial. For if $w_k \in \hat{r}^{-1} W' \hat{r}$, then $w_k = \hat{r}^{-1} w' \hat{r}$ for some $w' \in W'$ which would imply that $r = w'\hat{r} \in W'\hat{r}$, a contradiction. Letting $V_1 = N a_{\hat{r}} + (N a_{\hat{r}}) a_{w_k}$ and $V_2 = \sum_{d \in W'\backslash W/W_k, \text{ distinguished}; d \neq \hat{r}} N a_d A(W_k)$, Lemma (2.22) implies that

$$N \otimes_{A'} A = V_1 + V_2$$

is a decomposition into $A(W_k)$-bimodules. Thus

$$Z_M(A(W_k)) = Z_{V_1}(A(W_k)) + Z_{V_2}(A(W_k))$$

and since $m \in Z_M(A)$, we have that $n_{\hat{r}} a_{\hat{r}} + (n_{\hat{r}} a_{\hat{r}}) a_{w_k} \in Z_{V_1}(A(W_k))$. Thus we may apply the result of this theorem to V_1 since this is just the special case of a single fundamental reflection whose proof was given early in the proof of this theorem. This gives us

$$n_r a_r = (n_{\hat{r}} a_{\hat{r}}) a_{w_k} = u^{-1} a_{w_k} n_{\hat{r}} a_{\hat{r}} a_{w_k}.$$

By the minimality of r, we have

$$n_{\hat{r}} a_{\hat{r}} = u^{-l(\hat{r})} a_{\hat{r}-1} n_1 a_{\hat{r}}.$$

Hence

$$n_r a_r = u^{-1} a_{w_k} (u^{-l(\hat{r})} a_{\hat{r}-1} n_1 a_{\hat{r}}) a_{w_k} = u^{-l(r)} a_{r-1} n_1 a_r$$

after all. Therefore

$$m = \sum_{r \in R} u^{-l(r)} a_{r-1} n_1 a_r = N_{W,W'}(n_1),$$

and the theorem is proved. $\square$

Corollary (2.24). *Let M be an A-A bimodule and N an A-A' bisubmodule of M such that $M \simeq N \otimes_{A'} A$. Then*

$$Z_M(A) = N_{W,W'}(Z_M(A')).$$

Proof. By Proposition (2.13), we have $N_{W,W'}(Z_M(A')) \leq Z_M(A)$. Since $N \leq M$, we have

$$N_{W,W'}(Z_N(A)) \leq N_{W,W'}(Z_M(A')).$$

Theorem (2.23) then implies that

$$Z_M(A) = N_{W,W'}(Z_N(A')) \leq N_{W,W'}(Z_M(A')) \leq Z_M(A)$$

and the corollary follows. $\square$

Corollary (2.25). *Let S be a direct summand of an A-A bimodule $M = N \otimes_{A'} A$ with N an A-A' bimodule. Then*

$$N_{W,W'}(Z_S(A')) = Z_S(A).$$

Proof. Write $M = S \oplus (M - S)$. Theorem (2.23) implies that

$$N_{W,W'}(Z_M(A')) = Z_M(A).$$

From Proposition (2.13) we get

$$N_{W,W'}(Z_S(A')) \leq Z_S(A).$$

Let $s \in Z_S(A)$. Then $s \in Z_M(A) = N_{W,W'}(Z_M(A'))$. Write

$$s = \sum_{r \in R} u^{-l(r)} a_{r-1} m a_r$$

for some $m \in Z_M(A')$, where R is the set of distinguished right coset representatives for W' in W. Decompose m as $\hat{s} + t$ where $\hat{s} \in S$ and $t \in M - S$. Then

$$s = \sum_{r \in R} u^{-l(r)} a_{r-1} \hat{s} a_r + \sum_{r \in R} u^{-l(r)} a_{r-1} t a_r$$

and

$$s - \sum_{r \in R} u^{-l(r)} a_{r-1} \hat{s} a_r \in S.$$

But

$$\sum_{r \in R} u^{-l(r)} a_{r-1} t a_r \in M - S.$$

Thus

$$s - \sum_{r \in R} u^{-l(r)} a_{r-1} \hat{s} a_r \in S \cap (M - S) = \{0\}.$$

Hence

$$s = \sum_{r \in R} u^{-l(r)} a_{r-1} \hat{s} a_r \in N_{W,W'}(Z_S(A'))$$

from which the corollary is deduced. □

Theorem (2.26). *Let W_1 and W_2 be parabolic subgroups of W such that $W_2 = W_1^w$ for some $w \in W$. Let M be an A-A bimodule and N an A-A_1 bisubmodule of M such that $M = N \otimes_{A_1} A$. Then there is an A-A_2 bisubmodule N' of M such that*

$$N_{W,W_1}(Z_N(A_1)) = N_{W,W_2}(Z_{N'}(A_2)).$$

Proof. Let d be distinguished in the double coset $W_1 w W_2$ in W. Let N' be the A-A_2 bimodule $N \otimes_{A_1} a_d$. By Lemma (2.15) we have that $M \simeq N' \otimes_{A_2} A$. Applying Theorem (2.23) gives

$$N_{W,W_2}(Z_{N'}(A_2)) = Z_M(A).$$

But also we may apply Theorem (2.23) to $M = N \otimes_{A_1} A$ to get

$$N_{W,W_1}(Z_N(A_1)) = Z_M(A).$$

This proves the result. □

Corollary (2.27). *Given the same hypotheses as Theorem (2.26), we can conclude that*

$$N_{W,W_1}(Z_M(A_1)) = N_{W,W_2}(Z_M(A_2)).$$

Proof. Applying Corollary (2.26) to $N \otimes_{A_1} A$ we have

$$Z_M(A) = N_{W,W_1}(Z_M(A_1)).$$

From the proof of Theorem (2.26) we know that $M \simeq (N \otimes_{A_1} a_d) \otimes_{A_2} A$ where $d \in W_1 w W_2$ is distinguished. Applying Corollary (2.24) again gives

$$Z_M(A) = N_{W,W_2}(Z_M(A_2)). \quad □$$

Lemma (2.28). *Let W_1 and W_2 be parabolic subgroups of W and let D be the set of distinguished W_1-W_2 double coset representatives in W. Then*

$$A = \bigoplus_{d \in D} A_1 a_d A_2.$$

Proof. This follows immediately from the decomposition $W = \dot{\bigcup}_{d \in D} W_1 d W_2$.
□

Theorem (2.29) (Mackey decomposition). *Let W_1 and W_2 be parabolic subgroups of W and let D be the set of distinguished W_1-W_2 double coset representatives in W. Also, let N be an A-A_1 bimodule. Then*

$$N \otimes_{A_1} A \simeq \bigoplus_{d \in D} ((N \otimes_{A_1} a_d) \otimes_{A(W_1^d \cap W_2)} A_2).$$

Proof. From Lemma (2.28) we have $A = \bigoplus_{d \in D} A_1 a_d A_2$. By Corollary (2.20), $A_1 a_d A_2 \simeq A_1 a_d \otimes_{A(W_1^d \cap W_2)} A_2$. Thus

$$A \simeq \bigoplus_{d \in D} (A_1 a_d \otimes_{A(W_1^d \cap W_2)} A_2),$$

and hence

$$\begin{aligned}
N \otimes_{A_1} A &\simeq N \otimes_{A_1} \left(\bigoplus_{d \in D} A_1 a_d \otimes_{A(W_1^d \cap W_2)} A_2 \right) \\
&\simeq \bigoplus_{d \in D} \left(\left(N \otimes_{A_1} A_1 a_d \right) \otimes_{A(W_1^d \cap W_2)} A_2 \right) \\
&\simeq \bigoplus_{d \in D} \left(\left(N \otimes_{A_1} a_d \right) \otimes_{A(W_1^d \cap W_2)} A_2 \right). \quad \square
\end{aligned}$$

Theorem (2.30). *Let W_1 and W_2 be parabolic subgroups of W. Let R be the set of distinguished right coset representatives for W_1 in W. Let D be the set of distinguished W_1-W_2 double coset representatives in W. For $d \in D$, let T_d be the set of distinguished right coset representatives for $W_1^d \cap W_2$ in W_2. Let M be an A-A bimodule and let $b \in Z_M(A_1)$. Then*

(1) *for $d \in D$, $u^{-l(d)} a_{d^{-1}} b a_d \in Z_M(A(W_1^d \cap W_2))$.*

(2) *$N_{W,W_1}(b) = \sum_{d \in D} N_{W_2, W_1^d \cap W_2}(u^{-l(d)} a_{d^{-1}} b a_d)$.*

Proof. (1) Let $w \in W_1^d \cap W_2$ be a fundamental reflection. Then $w = d^{-1} w_1 d$ for some $w_1 \in W_1$. In other words, $dw = w_1 d$ and $l(d) + l(w) = l(dw) = l(w_1 d) = l(w_1) + l(d)$ by Lemma (2.17). Thus $l(w_1) = 1$ and so w_1 is fundamental. Hence,

$$\begin{aligned}
l(d^{-1} w_1) = l((d^{-1} w_1)^{-1}) &= l(w_1^{-1} d) \\
&= l(w_1 d) = l(w_1) + l(d) = l(w_1) + l(d^{-1}).
\end{aligned}$$

Similarly, $l(wd^{-1}) = l(w) + l(d^{-1})$. Therefore

$$\begin{aligned}
(a_{d^{-1}} b a_d) a_w = a_{d^{-1}} b a_{dw} &= a_{d^{-1}} b a_{w_1 d} \\
&= a_{d^{-1}} b a_{w_1} a_d = a_{d^{-1}} a_{w_1} b a_d \\
&= a_{d^{-1} w_1} b a_d = a_{wd^{-1}} b a_d = a_w (a_{d^{-1}} b a_d).
\end{aligned}$$

(2) Let $d \in D$ and $t \in T_d$. By Lemma (2.21), $dt \in R$ and $l(dt) = l(d) + l(t)$. Thus

$$\sum_{d \in D} N_{W_2, W_1^d \cap W_2}(u^{-l(d)} a_{d^{-1}} b a_d)$$

$$= \sum_{d \in D} \sum_{t \in T_d} u^{-l(t)} a_{t^{-1}} (u^{-l(d)} a_{d^{-1}} b a_d) a_t$$

$$= \sum_{d \in D} \sum_{t \in T_d} u^{-(l(t)+l(d))} a_{t^{-1}} a_{d^{-1}} b a_d a_t$$

$$= \sum_{d \in D} \sum_{t \in T_d} u^{-l(dt)} a_{(dt)^{-1}} b a_{dt}$$

$$= \sum_{r \in R} u^{-l(r)} a_{r^{-1}} b a_r \quad \text{since} \quad \sum_{d \in D} |W_1 d W_2| = |W|$$

$$\text{and} \quad |W_1 d W_2| = |W_1| \cdot [W_2 : W_1^d \cap W_2]$$

$$= N_{W, W_1}(b). \quad \square$$

Corollary (2.31). *Let M be an A-algebra (i.e., the action on M arises from an R-algebra homomorphism $A \to M$). Let W_1, W_2 and D be as in Theorem (2.30). Then*

$$N_{W, W_1}(Z_M(A_1)) \cdot N_{W, W_2}(Z_M(A_2)) \leq \sum_{d \in D} N_{W, W_1^d \cap W_2}(Z_M(A(W_1^d \cap W_2))).$$

Proof. Let $m_1 \in Z_M(A_1)$ and $m_2 \in Z_M(A_2)$. Then

$$N_{W, W_1}(m_1) \cdot N_{W, W_2}(m_2)$$

$$= N_{W, W_2}(N_{W, W_1}(m_1) m_2) \quad \text{by Proposition (2.13)}$$

$$= N_{W, W_2}\left(\sum_{d \in D} N_{W_2, W_1^d \cap W_2}(u^{-l(d)} a_{d^{-1}} m_1 a_d) m_2 \right) \quad \text{by Theorem (2.30)}$$

$$= N_{W, W_2}\left(\sum_{d \in D} N_{W_2, W_1^d \cap W_2}(u^{-l(d)} a_{d^{-1}} m_1 a_d \cdot m_2) \right) \quad \text{since } m_2 \in Z_M(A_2)$$

$$= \sum_{d \in D} N_{W, W_1^d \cap W_2}(u^{-l(d)} a_{d^{-1}} m_1 a_d \cdot m_2) \quad \text{by Lemma (2.12)}$$

which proves the corollary since $u^{-l(d)} a_{d^{-1}} m_1 a_d \cdot m_2 \in Z_M(A(W_1^d \cap W_2))$. $\quad \square$

Proposition (2.32). *Let W_1 and W_2 be parabolic subgroups of W. Let $\widehat{W}$ be a parabolic subgroup of W contained in $W_1 \times W_2$ so that $\widehat{W} = \widehat{W_1} \times \widehat{W_2}$ for some parabolic subgroups $\widehat{W_1}$ of W_1 and $\widehat{W_2}$ of W_2. Let $\hat{w} \in \widehat{W}$. Then*

$$N_{W_1 \times W_2, \widehat{W}}(a_{\hat{w}}) = N_{W_1, \widehat{W_1}}(a_{\hat{w}_1}) \cdot N_{W_2, \widehat{W_2}}(a_{\hat{w}_2}),$$

where $\hat{w} = \hat{w}_1 \hat{w}_2$ with $\hat{w}_1 \in \widehat{W_1}, \hat{w}_2 \in \widehat{W_2}$, and of course $l(\hat{w}_1 \hat{w}_2) = l(\hat{w}_1) + l(\hat{w}_2)$.

Proof. Clearly we may factor $\hat{w}$ as described in the statement of the proposition. Thus

$$
\begin{aligned}
N_{W_1 \times W_2, \hat{w}}(a_{\hat{w}}) &= N_{W_1 \times W_2, \widehat{W}_1 \times \widehat{W}_2}(a_{\hat{w}_1 \hat{w}_2}) \\
&= N_{W_1 \times W_2, W_1 \times \widehat{W}_2}(N_{W_1 \times \widehat{W}_2, \widehat{W}_1 \times \widehat{W}_2}(a_{\hat{w}_1 \hat{w}_2})) \quad \text{by Lemma (2.12)} \\
&= N_{W_2, \widehat{W}_2}(N_{W_1, \widehat{W}_1}(a_{\hat{w}_1 \hat{w}_2})) \\
&= N_{W_2, \widehat{W}_2}(N_{W_1, \widehat{W}_1}(a_{\hat{w}_1}) a_{\hat{w}_2}) \\
&= N_{W_1, \widehat{W}_1}(a_{\hat{w}_1}) \cdot N_{W_2, \widehat{W}_2}(a_{\hat{w}_2}). \quad \square
\end{aligned}
$$

Definition (2.33). A right A-module M is A'-projective if every exact diagram of right A-modules and maps which can be completed to a commutative diagram over A' can also be completed to a commutative diagram over A:

$$
\begin{array}{ccc}
& & M \\
& \nearrow & \downarrow \\
U & \longrightarrow V & \longrightarrow 0
\end{array}
$$

Theorem (2.34) ("Higman's Criterion" for A-modules). *Let M be a right A-module. Then the following are equivalent*:

(1) M *is* A'-*projective.*
(2) $M \mid M \otimes_{A'} A$; *that is, M is isomorphic to a direct summand of $M \otimes_{A'} A$.*
(3) $M \mid U \otimes_{A'} A$ *for some right A'-module U.*
(4) $M_{W,W'}(\mathrm{Hom}_{A'}(M,M)) = \mathrm{Hom}_A(M,M)$.

Proof. $(1) \Rightarrow (2)$ Let $f: M \otimes_{A'} A \to M$ be defined as $f(m \otimes a) = ma$. This is surjective and we can complete the following diagram to make it a commutative A'-diagram:

$$
\begin{array}{ccc}
& & M \\
& \nearrow & \downarrow {}^{1_M} \\
M \otimes_{A'} A & \underset{f}{\longrightarrow} M & \longrightarrow 0
\end{array}
$$

So by (1) this can be completed to a commutative diagram over A, say by $\hat{\theta}$. Thus $f\hat{\theta} = 1_M$ and $M \mid M \otimes_{A'} A$.

  $(2) \Rightarrow (3)$ This is obvious.
  $(3) \Rightarrow (4)$ Define $g: U \otimes_{A'} A \to U \otimes_{A'} A$ by

$$
\left(\sum_{i \in R} u_i \otimes a_i \right) g = u_1 \otimes a_1;
$$

R = set of distinguished right coset
representatives for W' in W.

Then for $a' \in A'$,

$$\left(\left(\sum_{i \in R} u_i \otimes a_i\right) a'\right) g = \left(\sum_{i \in R} u_i \otimes a_i a'\right) g$$

$$= u_1 \otimes a' = (u_1 \otimes a_1)a' = \left(\sum_{i \in R} u_i \otimes a_i\right) g \cdot a'$$

so that g is an A'-map.

Let $v = \sum_{i \in R} u_i \otimes a_i \in U \otimes_{A'} A$. Then

$$\left(\sum_{i \in R} u_i \otimes a_i\right) N_{W,W'}(g) = \left(\sum_{i \in R} u_i \otimes a_i\right) \left(\sum_{r \in R} u^{-l(r)} a_{r-1} g a_r\right)$$

$$= \sum_{r \in R} (u_r \otimes a_1)a_r = \sum_{r \in R} (u_r \otimes a_r) = v$$

so that $N_{W,W'}(g) = 1_{U \otimes_{A'} A}$ =identity on $U \otimes_{A'} A$.

Let h be the projection of $U \otimes_{A'} A$ onto M. Clearly h is an A-map. Thus the restriction of gh to M, denoted $gh\,|_M$, is an element of $\mathrm{Hom}_{A'}(M, M)$. Then for $m \in M$,

$$(m)N_{W,W'}(gh\,|_M) = m,$$

and so

$$1_M = N_{W,W'}(gh\,|_M) \in N_{W,W'}(\mathrm{Hom}_{A'}(M, M))$$

which implies (4).

(4) $\Rightarrow$ (1) There exists $\psi \in \mathrm{Hom}_{A'}(M, M)$ such that $N_{W,W'}(\psi) = 1_M$. Assume the following exact diagram of right A-modules can be completed by θ to a commutative diagram over A':

$$\begin{array}{ccc} & & M \\ & \overset{\theta}{\nearrow} & \downarrow \\ U \searrow & \xrightarrow{\quad f \quad} & V \longrightarrow 0 \end{array}$$

Then

$$N_{W,W'}(\psi\theta)f = N_{W,W'}(\psi\theta f) = N_{W,W'}(\psi g) = N_{W,W'}(\psi)g = g.$$

Since $N_{W,W'}(\psi\theta)$ is an A-map by Proposition (2.13), (1) follows and this completes the proof of the theorem. $\square$

Remark. Theorem (2.34) also holds on either side of an A-A bimodule M.

Theorem (2.35). *Let Z be a finitely generated indecomposable right A-module. Then there exists a parabolic subgroup W' of W unique up to conjugation such that Z is A'-projective and such that W' is W-conjugate to a parabolic subgroup of any parabolic subgroup W'' of W for which Z is A''-projective.*

Proof. Let $V(Z)$ be the set of all subgroups $\widehat{W}$ of W for which Z is $A(\widehat{W})$-projective. Let W' be a minimal element of $V(Z)$ with respect to the partial order containment. Let W'' be any element of $V(Z)$. Write $Z = \bigoplus_{i=1}^{t} U_i$

with each U_i an indecomposable right A'-module. Then $Z | Z \otimes_{A'} A$ $= \bigoplus_{i=1}^{t} (U_i \otimes_{A'} A)$ and $Z | U_i \otimes_{A'} A$ for some i by the Krull-Schmidt Theorem [9] (with suitable hypotheses on R). We also have that $Z | Z \otimes_{A''} A$. Theorem (2.29) implies that

$$Z \otimes_{A''} A = \bigoplus_{d \in D} ((Z \otimes_{A''} a_d) \otimes_{A((W'')^d \cap W')} A'),$$

where D is the set of distinguished W''-W' double coset representatives in W. Thus $Z | (Z \otimes_{A''} a_d) \otimes_{A((W'')^d \cap W')} A'$ for some d and so

$$U_i \mid (Z \otimes_{A''} a_d) \otimes_{A((W'')^d \cap W')} A'$$

for this d. Then $U_i \otimes_{A'} A \mid (Z \otimes_{A''} a_d) \otimes_{A((W'')^d \cap W')} A$, and hence

$$Z | (Z \otimes_{A''} a_d) \otimes_{A((W'')^d \cap W')} A.$$

That is, Z is $A((W'')^d \cap W')$-projective. But W' is minimal, so $W' \le (W'')^d$. Thus $(W')^{d^{-1}} \le W''$, which proves the theorem. $\square$

3. A basis for $Z(A(S_n))$ over $\mathbf{Q}[u, u^{-1}]$

Recall from §1 that S_n is the Weyl group for $\mathrm{SL}(n, q)$ and that S_n is generated by the set $\{(12), (23), \dots, (n-1\, n)\}$ of fundamental reflections. In this section we obtain a basis for the center of $A(S_n)$ over $\mathbf{Q}[u, u^{-1}]$. This basis consists of relative norms of elements in certain parabolic subgroups of S_n, *normed up* from these subgroups to S_n. A particular element in $A(S_n)$ of special importance to this construction is given in the following definition.

Definition (3.1). The element η in $A(S_n)$ is defined as

$$\eta = N_{S_{n-1}, S_1}(a_{(12)(23)\cdots(n-1\, n)}),$$

with $S_{n-1} = \langle (12), (23), \dots, (n-2\, n-1) \rangle$.

Note. For $r \le n-2$, S_{n-r} will mean $\langle (12), (23), \dots, (n-r-1\, n-r) \rangle$, unless otherwise noted.

Lemma (3.2). $\eta \in Z(A(S_n))$.

Proof. For $n = 2$ this is clearly true. Proceed by induction. By Proposition (3.13) we have immediately that $\eta \in Z_{A(S_n)}(A(S_{n-1}))$, so to prove this lemma, it will be enough to show that

$$(3.3) \qquad\qquad a_{(n-1\, n)}\eta = \eta a_{(n-1\, n)}.$$

To do this, we will examine in further detail the element η. Let

$$\alpha = N_{S_{n-2}, S_1}(a_{(12)(23)\cdots(n-2\, n-1)}).$$

Then

$$\eta = N_{S_{n-1},S_{n-2}}(N_{S_{n-2},S_1}(a_{(12)(23)\cdots(n-1\,n)})), \quad \text{by Lemma (2.12)}$$
$$= N_{S_{n-1},S_{n-2}}(\alpha a_{(n-1\,n)}), \quad \text{since } a_{(n-1\,n)} \in Z_{A(S_n)}(A(S_{n-2}))$$
$$= \alpha N_{S_{n-1},S_{n-2}}(a_{(n-1\,n)}), \quad \text{by induction, } \alpha \in Z(A(S_{n-1}))$$
$$= N_{S_{n-1},S_{n-2}}(a_{(n-1\,n)})\alpha, \quad \text{by Proposition (2.13)},$$
$$N_{S_{n-1},S_{n-2}}(a_{(n-1\,n)}) \in Z_{A(S_n)}(A(S_{n-1})).$$

Hence, (3.3) holds if and only if

$$(3.4) \qquad a_{(n-1\,n)}\alpha N_{S_{n-1},S_{n-2}}(a_{(n-1\,n)}) = N_{S_{n-1},S_{n-2}}(a_{(n-1\,n)})\alpha a_{(n-1\,n)}.$$

From Theorem (2.30), part (2), with $W = S_{n-1}$ and $W_1 = W_2 = S_{n-2}$, we have that

$$N_{S_{n-1},S_{n-2}}(a_{(n-1\,n)}) = a_{(n-1\,n)} + u^{-1}N_{S_{n-2},S_{n-3}}(a_{(n-2\,n-1)(n-1\,n)(n-2\,n-1)}).$$

Replacing $N_{S_{n-1},S_{n-2}}(a_{(n-1\,n)})$ with this expression in equation (3.4), we get a common term of $a_{(n-1\,n)}\alpha a_{(n-1\,n)}$ on both sides of the equation. Thus, by subtracting this common term and multiplying both sides by u, we have that (3.4) holds if and only if

$$(3.5) \qquad a_{(n-1\,n)}\alpha N_{S_{n-2},S_{n-3}}(a_{(n-2\,n-1)(n-1\,n)(n-2\,n-1)})$$
$$= N_{S_{n-2},S_{n-3}}(a_{(n-2\,n-1)(n-1\,n)(n-2\,n-1)})\alpha a_{(n-1\,n)}.$$

Rewriting $a_{(n-2\,n-1)(n-1\,n)(n-2\,n-1)}$ as $a_{(n-1\,n)(n-2\,n-1)(n-1\,n)}$ using the relations in S_n and recalling that $a_{(n-1\,n)} \in Z_{A(S_n)}(A(S_{n-2}))$ gives us that equation (3.5) is true if and only if

$$(3.6) \qquad a_{(n-1\,n)}\alpha a_{(n-1\,n)}N_{S_{n-2},S_{n-3}}(a_{(n-2\,n-1)})a_{(n-1\,n)}$$
$$= a_{(n-1\,n)}N_{S_{n-2},S_{n-3}}(a_{(n-2\,n-1)})a_{(n-1\,n)}\alpha a_{(n-1\,n)}.$$

Now using Lemma (2.12) and the fact that $a_{(n-2\,n-1)} \in Z_{A(S_{n-1})}(A(S_{n-3}))$, rewrite α as

$$N_{S_{n-2},S_{n-3}}(N_{S_{n-3},S_1}(a_{(12)(23)\cdots(n-3\,n-2)})a_{(n-2\,n-1)}).$$

By induction, $N_{S_{n-3},S_1}(a_{(12)(23)\cdots(n-3\,n-2)}) \in Z(A(S_{n-2}))$, and so

$$\alpha = N_{S_{n-3},S_1}(a_{(12)(23)\cdots(n-3\,n-2)}) \cdot N_{S_{n-2},S_{n-3}}(a_{(n-2\,n-1)}).$$

This expression for α allows us to rewrite the left-hand side of equation (3.6) as

$$(3.7) \qquad a_{(n-1\,n)}N_{S_{n-3},S_1}(a_{(12)(23)\cdots(n-3\,n-2)})$$
$$\cdot N_{S_{n-2},S_{n-3}}(a_{(n-2\,n-1)})a_{(n-1\,n)}$$
$$\cdot N_{S_{n-2},S_{n-3}}(a_{(n-2\,n-1)})a_{(n-1\,n)}.$$

But $N_{S_{n-3},S_1}(a_{(12)(23)\cdots(n-3\,n-2)}) \in A(S_{n-2})$ and by Proposition (2.13),

$$N_{S_{n-2},S_{n-3}}(a_{(n-2\,n-1)}) \in Z_{A(S_{n-1})}(A(S_{n-2})).$$

Thus, recalling once again that $a_{(n-1\,n)} \in Z_{A(S_n)}(A(S_{n-2}))$, (3.7) may be rearranged to get

$$a_{(n-1\,n)}N_{S_{n-2},S_{n-3}}(a_{(n-2\,n-1)})a_{(n-1\,n)}$$
$$\cdot N_{S_{n-3},S_1}(a_{(12)(23)\cdots(n-3\,n-2)})N_{S_{n-2},S_{n-3}}(a_{(n-2\,n-1)})a_{(n-1\,n)}$$
$$= a_{(n-1\,n)}N_{S_{n-2},S_{n-3}}(a_{(n-2\,n-1)})a_{(n-1\,n)}\alpha a_{(n-1\,n)}$$

which is the right-hand side of equations (3.6), and the lemma is proved. □

Remark. We would like to be able to define η as $N_{S_n,\langle(12\cdots n)\rangle}(a_{(12)(23)\cdots(n-1\,n)})$ to avoid the need for Lemma (3.2), but this is not permitted under our definition if the norm since $\langle(12\cdots n)\rangle$ is not a parabolic subgroup of S_n. Nevertheless, $\langle(12\cdots n)\rangle$ is, in some sense, close to being a parabolic subgroup. Since $S_n = \langle(12\cdots n)\rangle S_{n-1}$, we see that $\langle(12\cdots n)\rangle$ is like the *difference* of the two parabolic subgroups S_n and S_{n-1}. Because of this, one might label $\langle(12\cdots n)\rangle$ a *virtual* parabolic subgroup.

It will prove convenient to have various representations for the element η. In the next lemma we give some useful forms of η.

Lemma (3.8). *The following are equal:*

(1) η;

(2) $N_{S_{n-1},S_{n-2}}(N_{S_{n-2},S_1}(a_{(12)(23)\cdots(n-2\,n-1)})a_{(n-1\,n)})$;

(3) $N_{S_{n-2},S_1}(a_{(12)(23)\cdots(n-2\,n-1)})N_{S_{n-1},S_{n-2}}(a_{(n-1\,n)})$;

(4) $a_{(12)}N_{S_2,S_1}(a_{(23)})N_{S_3,S_2}(a_{(34)})\cdots N_{S_{n-1},S_{n-2}}(a_{(n-1\,n)})$;

(5) $a_{(12)}(a_{(23)}+u^{-1}a_{(13)})(a_{(34)}+u^{-1}a_{(24)}+u^{-2}a_{(14)})\cdots(a_{(n-1\,n)}+u^{-1}a_{(n-2\,n)}+$
$\cdots+u^{-(n-2)}a_{(1n)})$.

Proof. Using Lemma (2.12) and the fact that $a_{(n-1\,n)} \in Z_{A(S_n)}(A(S_{n-2}))$, (2) can be derived from (1). Lemma (3.2) implies that $N_{S_{n-2},S_1}(a_{(12)(23)\cdots(n-2\,n-1)}) \in Z(A(S_{n-1}))$ so that (3) equals (2). Number (4) follows from (3) by induction, and (5) is just an expansion of (4). □

Remark. Adopting the convention that $S_0 = S_1$ =identity subgroup, representation (4) in Lemma (3.8) may be written as

$$(3.9) \qquad \prod_{i=1}^{n-1} N_{S_i,S_{i-1}}(a_{(i\,i+1)}).$$

By Proposition (2.13), $N_{S_i,S_{i-1}}(a_{(i\,i+1)}) \in Z_{A(S_{i+1})}(A(S_i))$. This implies that each factor in (3.9) commutes with every other factor in (3.9). Thus,

$$\eta = N_{S_{n-1},S_{n-2}}(a_{(n-1\,n)})N_{S_{n-2},S_{n-3}}(a_{(n-2\,n-1)})\cdots N_{S_2,S_1}(a_{(23)})a_{(12)}.$$

Then an easy induction argument can be used to deduce that

$$N_{S_{n-1},S_{n-2}}(a_{(n-1\,n)})N_{S_{n-2},S_{n-3}}(a_{(n-2\,n-1)})\cdots N_{S_2,S_1}(a_{(23)})a_{(12)}$$
$$= N_{S_{n-1},S_1}(a_{(n-1\,n)(n-2\,n-1)\cdots(23)(12)})\,.$$

Hence, $\eta = N_{S_{n-1},S_1}(a_{(n-1\,n)(n-2\,n-1)\cdots(23)(12)})$. This representation of η will be used to prove a more general statement about η in Lemma (3.16).

The following lemma, although trivial, contains observations which are helpful in giving clarity to later results.

Lemma (3.10). *Let $\alpha \in S_n$ be an n-cycle. Then*

(1) *each reduced expression for α must involve all of the fundamental reflections of S_n,*

(2) *$l(\alpha) \geq n-1$,*

(3) *if $l(\alpha) = n-1$, each reduced expression for α involves each fundamental reflection of S_n exactly once,*

(4) *if $l(\alpha) = n-1$, there is only one reduced expression for α up to rearrangement of the commuting fundamental reflections in α.*

Proof. If a reduced expression for α is missing a fundamental reflection of S_n, then α, an n-cycle, would be living inside a parabolic subgroup of S_n. This is impossible since proper parabolic subgroups of S_n are direct products of symmetric groups S_i where $i < n$. This proves (1). Parts (2) and (3) follow easily from (1), and the fact that $S_n = \langle (12), (23), \ldots, (n-1\,n) \rangle$. Part (4) is an immediate consequence of (3).

Lemma (3.11). *Let $\alpha \in s_n$ with $l(\alpha) = n-1$. Suppose that a reduced expression for α involves every fundamental reflection of S_n exactly once. Then α is an n-cycle.*

Proof. Assume without loss of generality that the fundamental reflection $(n-1\,n)$ appears to the right of $(n-2\,n-1)$ in the reduced expression for α. Since $(n-1\,n)$ commutes with every other fundamental reflection of S_n, we may write

$$\alpha = \alpha'(n-1\,n)$$

with $\alpha' \in S_{n-1} = \langle (12), (23), \ldots, (n-2\,n-1) \rangle$, α' reduced, $l(\alpha') = n-2$ and α' involving every fundamental reflection of S_{n-1} exactly once. By induction, α' is an $(n-1)$-cycle. Write

$$\alpha' = (\pi_1 \cdots n-1 \cdots \pi_{n-1})\,.$$

Then

$$\alpha = (\pi_1 \cdots n-1 \cdots \pi_{n-1})(n-1\,n)$$
$$= (\pi_1 \cdots n-1\,n \cdots \pi_{n-1})$$

is an n-cycle. $\square$

Remark. Although there are redundancies in the hypotheses of Lemma (3.11), we have stated it this way for clarity.

Definition (3.12). A partition of n is a finite sequence of positive integers $\alpha = (k_1, k_2, \ldots, k_t)$ such that
(1) $k_i \geq k_{i+1}$ for all i,
(2) $\sum_i k_i = n$.
The fact that α is a partition of n is abbreviated $\alpha \vdash n$.

If $\pi \in S_n$, then the ordered lengths of the cyclic factors of π in cycle notation form a uniquely determined partition of n. Since cycle structure in S_n uniquely determines conjugacy class, we have a one-to-one correspondence, in a natural way, between partitions of n and conjugacy classes of S_n.

Lemma (3.13). *Every conjugacy class in S_n, with the exception of the class of n-cycles, contains an element of length less than $n - 1$.*

Proof. Let $\alpha \vdash n$ with $\alpha = (k_1, k_2, \ldots, k_t)$, $t > 1$, so that α corresponds to a conjugacy class C_α of S_n other than the class of n-cycles. Consider the element

$$
\begin{aligned}
\pi &= (12)(23) \cdots (k_1 - 1\, k_1)(k_1 + 1\, k_1 + 2)(k_1 + 2\, k_1 + 3) \\
&\quad \cdots (k_1 + k_2 - 1\, k_1 + k_2) \\
&\quad \cdots (k_1 + k_2 + \cdots + k_{t-1} + 1\, k_1 + k_2 + \cdots + k_{t-1} + 2) \\
&\quad \cdots (k_1 + k_2 + \cdots + k_t - 1\, k_1 + k_2 + \cdots + k_t) \\
&= (123 \cdots k_1)(k_1 + 1\, k_1 + 2 \cdots k_1 + k_2) \\
&\quad \cdots (k_1 + k_2 + \cdots k_{t-1} + 1 \cdots k_1 + k_2 + \cdots + k_t),
\end{aligned}
$$

where the convention is that $(r\,r)$ is the identity. Clearly $\pi \in C_\alpha$ and $l(\pi) = \sum_{i=1}^{t}(k_i - 1) = \sum_{i=1}^{t} k_i - t = n - t < n - 1$. The lemma follows from Lemma (3.10), part (2). $\square$

Notes. (1) When we say an element a_γ in $A(S_n)$ contains or involves a particular fundamental reflection, we mean that any reduced expression for γ contains or involves that particular fundamental reflection in S_n.

(2) Given $a_\gamma \in A(S_n)$, the length of a_γ, denoted $l(a_\gamma)$, will mean the length of γ, $l(\gamma)$.

(3) Let $\alpha \in A(S_n)$. Then α may be written as

$$
\alpha = \sum_{w \in S_n} f_w(u) a_w \quad \text{with } f_w(u) \in \mathbf{Q}[u, u^{-1}].
$$

By a term of α we mean $f_w(u) a_w$ for some $w \in S_n$, with $f_w(u) \neq 0$.

Lemma (3.14). *Every term of η contains each of the fundamental reflections $(12), (23), \ldots, (n - 1\, n)$.*

Proof. Appealing to Lemma (3.8), part (3), and expanding we have

$$
\begin{aligned}
\eta &= N_{S_{n-2}, S_1}(a_{(12)(23)\cdots(n-2\, n-1)}) N_{S_{n-1}, S_{n-2}}(a_{(n-1\, n)}) \\
&= N_{S_{n-2}, S_1}(a_{(12)(23)\cdots(n-2\, n-1)}) \sum_{t=1}^{n-1} u^{-(l((n-t\, n))-1)/2} a_{(n-t\, n)}.
\end{aligned}
$$

By induction, every term of $N_{S_{n-2}, S_1}(a_{(12)(23)\cdots(n-2\,n-1)})$ contains each of the fundamental reflections $(12), (23), \ldots, (n-2\,n-1)$. Let a_γ be an arbitrary term of $N_{S_{n-2}, S_1}(a_{(12)(23)\cdots(n-2\,n-1)})$, disregarding the coefficient. Let $\tau = (n-1\,n)(n-2\,n-1)\cdots(n-t\,n-t+1)$ for a fixed t with $1 \le t \le n-1$. Then

$$a_\gamma \cdot a_{(n-t\,n)} = a_\gamma a_{(n-t\,n-t+1)(n-t+1\,n-t+2)\cdots(n-2\,n-1)} a_\tau.$$

Each of the terms $f_\lambda(u)a_\lambda$, $f_\lambda(u) \neq 0$, of the product

$$a_\gamma a_{(n-t\,n-t+1)(n-t+1\,n-t+2)\cdots(n-2\,n-1)}$$

is such that $\lambda \in S_{n-1}$ with λ involving all of the fundamental reflections of S_{n-1} with the possible exception of $(n-t\,n-t+1)$, $(n-t+1\,n-t+2), \ldots,$ $(n-2\,n-1)$. But τ contains these possible deficiencies in addition to $(n-1\,n)$, and since τ is a distinguished right coset representative for S_{n-1} in S_n, we have by Proposition (1.1.2) that $a_\lambda a_\tau = a_{\lambda\tau}$ so that every term in $a_\gamma a_{(n-t\,n)}$ contains each of the fundamental reflections $(12), (23), \ldots, (n-1\,n)$. The lemma follows. $\square$

Corollary (3.15). *Let $f_\gamma(u)a_\gamma$ be an arbitrary term in η with $f_\gamma(u) \neq 0$. Then $l(\gamma) \ge n-1$.*

Proof. Since there are $n-1$ fundamental reflections of S_n, the corollary is an immediate consequence of Lemma (3.14). $\square$

Lemma (3.16). *Let $\tau \in S_n$ be an n-cycle with $l(\tau) = n-1$. Then*

$$N_{S_{n-1}, S_1}(a_\tau) = \eta.$$

Proof. By Lemma (3.10), part (3), a reduced expression for τ involves every fundamental reflection of S_n exactly once. Assume first that $(n-1\,n)$ occurs to the right of $(n-2\,n-1)$ in a reduced expression for τ. Since $(n-1\,n)$ commutes with every other fundamental reflection of S_n, we may write

$$\tau = \tau'(n-1\,n),$$

where τ' is reduced, $l(\tau') = n-2$ and τ' involves each of the fundamental reflections $(12), (23), \ldots, (n-2\,n-1)$ exactly once. Thus Lemma (3.11) implies that τ' is an $(n-1)$-cycle. Therefore

$$
\begin{aligned}
N_{S_{n-1}, S_1}(a_\tau) &= N_{S_{n-1}, S_1}(a_{\tau'} a_{(n-1\,n)}) \\
&= N_{S_{n-1}, S_{n-2}}(N_{S_{n-2}, S_1}(a_{\tau'})a_{(n-1\,n)}).
\end{aligned}
$$

Since this lemma is trivial for $n = 2$, by induction we have $N_{S_{n-2}, S_1}(a_{\tau'}) = N_{S_{n-2}, S_1}(a_{(12)(23)\cdots(n-2\,n-1)})$. Hence

$$
\begin{aligned}
N_{S_{n-1}, S_1}(a_\tau) &= N_{S_{n-1}, S_{n-2}}(N_{S_{n-2}, S_1}(a_{(12)(23)\cdots(n-2\,n-1)})a_{(n-1\,n)}) \\
&= \eta \quad \text{by Lemma (3.8).}
\end{aligned}
$$

Now assume that $(n - 1\, n)$ occurs to the left of $(n - 2\, n - 1)$ in a reduced expression for τ. Following the same line of reasoning as above, we may write

$$\tau = (n - 1\, n)\tau',$$

where τ' is reduced, $l(t') = n - 2$ and τ' involves each of the fundamental reflections $(12), (23), \ldots, (n - 2\, n - 1)$ exactly once. As before, Lemma (3.11) implies τ' is an $(n - 1)$-cycle and we have

$$N_{S_{n-1}, S_1}(a_\tau) = N_{S_{n-1}, S_1}(a_{(n-1\, n)} a_{\tau'})$$
$$= N_{S_{n-1}, S_{n-2}}(a_{(n-1\, n)} N_{S_{n-2}, S_1}(a_{\tau'})).$$

By induction we have

$$N_{S_{n-2}, S_1}(a_{\tau'}) = N_{S_{n-2}, S_1}(a_{(n-2\, n-1)(n-3\, n-2)\cdots(23)(12)}).$$

Thus

$$N_{S_{n-1}, S_1}(a_\tau) = N_{S_{n-1}, S_{n-2}}(a_{(n-1\, n)} N_{S_{n-2}, S_1}(a_{(n-2\, n-1)\cdots(12)}))$$
$$= N_{S_{n-1}, S_{n-2}}(N_{S_{n-2}, S_1}(a_{(n-1\, n)(n-2\, n-1)\cdots(12)})$$
$$= N_{S_{n-1}, S_1}(a_{(n-1\, n)(n-2\, n-1)\cdots(23)(12)})$$
$$= \eta$$

by the remark following Lemma (3.8).   □

Lemma (3.17). *Let* $f_\gamma(u)a_\gamma$ *be a term of* η *with* $f_\gamma(u) \neq 0$. *Suppose* $l(\gamma) = n - 1$. *Then* γ *is an* n-cycle.

Proof. In taking the norm of $a_{(12)(23)\cdots(n-1\, n)}$ to get η, the lengths for the right multiplications add by Proposition (1.1.2). Hence, the only way to get back down to an element a_γ, with $l(\gamma) = n - 1$, is to have every fundamental reflection coming from the corresponding element being multiplied on the left reduce the length. That is, γ must be of the form $r^{-1}(12)(23) \cdots (n - 1\, n)r$ which is of course an n-cycle.   □

Lemma (3.18). *Let* $\pi \in S_n$ *be an* n-cycle with $l(\pi) = n - 1$. *Then* a_π *occurs with coefficient one in the complete expansion of* η.

Proof. Lemma (3.10), part (3), implies that $(n - 1\, n)$ occurs in any reduced expression for π. Assume that $(n - 1\, n)$ occurs to the right of $(n - 2\, n - 1)$ in a reduced expression for π. So we may write

$$\pi = \pi'(n - 1\, n),$$

where π' is reduced, $l(\pi') = n - 2$ and π' involves each of the fundamental reflections $(12), (23), \ldots, (n - 2\, n - 1)$ exactly once. Hence, by Lemma (3.11),

π' is an $(n-1)$-cycle. Then using Lemmas (3.8) and (3.16), we get

$$
\begin{aligned}
\eta &= N_{S_{n-1},S_{n-2}}(N_{S_{n-2},S_1}(a_{\pi'})a_{(n-1\,n)}) \\
&= N_{S_{n-1},S_{n-2}}(N_{S_{n-2},S_1}(a_{(12)(23)\cdots(n-2\,n-1)})a_{(n-1\,n)}) \\
&= N_{S_{n-2},S_1}(a_{(12)(23)\cdots(n-2\,n-1)})N_{S_{n-1},S_{n-2}}(a_{(n-1\,n)}) \\
&= N_{S_{n-2},S_1}(a_{(12)(23)\cdots(n-2\,n-1)})\sum_{t=1}^{n-1}u^{-(l((t\,n))-1)/2}a_{(t\,n)}.
\end{aligned}
$$

From this we see that to get the term a_π there must be some $\gamma \in S_{n-1}$ such that

$$
\gamma(k\,n) = \pi = \pi'(n-1\,n) \quad \text{for some } 1 \le k \le n-1.
$$

This implies that

$$
(k\,n)(n-1\,n) = \gamma^{-1}\pi' \in S_{n-1}
$$

which is a contradiction unless $k = n-1$. Thus $\gamma = \pi'$. This lemma holds obviously for $n = 2$ and so we assume by induction that $a_{\pi'}$ occurs with coefficient one in $N_{S_{n-2},S_1}(a_{(12)(23)\cdots(n-2\,n-1)})$. This forces the coefficient of a_π to be one in η. The case when $(n-1\,n)$ occurs to the left of $(n-2\,n-1)$ in a reduced expression for π is handled in a similar manner. □

Notation. Let $\alpha \in A(S_n)$. We will write $\alpha\,|_{u=1}$ to mean the *image* of α in the specialization of $A(S_n)$ at $u = 1$. We say that α is specialized at $u = 1$.

Lemma (3.19). *Utilizing the fact that* $A(S_n)\,|_{u=1} \simeq \mathbf{Q}S_n$, *we have that* $\eta|_{u=1}$ *equals the sum of the n-cycles in* $\mathbf{Q}S_n$, *making the natural identification of* a_w *in* $A(S_n)$ *with* w *in* S_n.

Proof. Since $\eta\,|_{u=1} \in Z(\mathbf{Q}S_n)$, it may be written as a **Q**-linear combination of conjugacy class sums for S_n. Using Lemma (3.13) and Corollary (3.15) we conclude that $\eta\,|_{u=1}$ must be a **Q**-multiple of the single class sum of n-cycles. By Lemma (3.18), this multiple is one and the lemma is proved. □

Remark. Lemma (3.19) follows easily from the observation that

$$
\begin{aligned}
\eta\,|_{u=1} &= N_{S_{n-1},S_1}(a_{(12)(23)\cdots(n-1\,n)})\,|_{u=1} \\
&= N_{S_{n-1},S_1}((12)(23)\cdots(n-1\,n)),
\end{aligned}
$$

which is the class sum of n-cycles in S_n. The proof given above is more consistent with the general methods of proof given for other results in this section.

Definition (3.20). For each partition $\alpha = (k_1, k_2, \ldots, k_t)$ of n (recall $k_1 \ge k_2 \ge \cdots \ge k_t$), we say a parabolic subgroup $P_{\alpha,s}$ of S_n corresponds to α if

$$
P_{\alpha,s} = S_{k_1} \times S_{k_2} \times \cdots \times S_{k_t},
$$

where s is a numeric subscript used to distinguish among the different parabolic subgroups of S_n having the shape of α. For example, for $n = 5$, $P_{\alpha,1} =$

$\langle(12)\rangle \times \langle(34)\rangle \times S_1$ and $P_{\alpha,2} = \langle(23)\rangle \times \langle(45)\rangle \times S_1$ both have shape $\alpha = (2,2,1)$. In some sense there is a natural representative parabolic subgroup to choose from among all parabolic subgroups having shape α, namely,

$$S_{k_1} \times S_{k_2} \times \cdots \times S_{k_t}$$

with

$$S_{k_1} = \langle(12),(23),\ldots,(k_1 - 1\; k_1)\rangle,$$
$$S_{k_2} = \langle(k_1 + 1\; k_1 + 2),(k_1 + 2\; k_1 + 3),\ldots,(k_1 + k_2 - 1\; k_1 + k_2)\rangle$$

and so on. We will let $P_{\alpha,1}$ be this particular *left-justified* choice and denote it simply as P_α, deleting the subscript s.

Definition (3.21). Let $P_{\alpha,s}$ be a parabolic subgroup of S_n corresponding to a partition $\alpha = (k_1, k_2, \ldots, k_t)$ of n as in Definition (3.20). Define

$$\eta_{\alpha,s} = \prod_{i=1}^{t} N_{S_{k_i-1},S_1}(a_{w_i}),$$

$$l_\alpha = \sum_{i=1}^{t}(k_i - 1),$$

where s is as it was in Definition (3.20), and where w_i is a k_i-cycle in S_{k_i} of length $k_i - 1$ and $N_{S_{k_i-1},S_1}(a_{w_i}) = a_1$ if $k_i = 1$. Also, $\eta_\alpha = \eta_{\alpha,1}$.

Remarks. (1) For $\alpha = (n)$, there is of course only one $P_{\alpha,s}$, namely $P_\alpha = S_n$. So then $\eta_{\alpha,s} = \eta_\alpha = \eta$.

(2) If a fundamental reflection appears in one of the norm factors of $\eta_{\alpha,s}$ it will not appear in any other factor. This implies that these factors commute with each other and the lengths add in the multiplications.

Lemma (3.22). *Let* $\alpha, \beta \in A(S_n)$. *Then*
  (1) $(\alpha + \beta)|_{u=1} = \alpha|_{u=1} + \beta|_{u=1}$,
  (2) $(\alpha\beta)|_{u=1} = \alpha|_{u=1} \cdot \beta|_{u=1}$,

Proof. (1) This is clear.

(2) In view of (1), we only need to consider the trivial case $\alpha = f(u)a_w$, $\beta = g(u)a_{w'}$, where $w, w' \in S_n$. Then

$$\alpha\beta|_{u=1} = f(u)g(u)a_w a_{w'}|_{u=1}$$
$$= f(u)g(u)(u^{\bar{z}}a_{ww'} + (u-1)(\text{ possible other terms}))|_{u=1}$$
$$= f(1)g(1)a_{ww'} = f(1)a_w g(1)a_{w'} = \alpha|_{u=1} \cdot \beta|_{u=1}. \quad \square$$

Lemma (3.23). *Let* $\alpha, P_{\alpha,s}, \eta_{\alpha,s}$ *and* l_α *be as in Definition* (3.21). *Then*
  (1) $\eta_{\alpha,s}$ *involves only terms of length* l_α *and longer,*
  (2) *the coefficient on all terms corresponding to elements in the class of* α *of length* l_α *is one,*
  (3) $\eta_{\alpha,s} \in Z(A(P_{\alpha,s}))$,

(4) $\eta_{\alpha,s}|_{u=1} = \prod_{i=1}^{t}$ (sum of the k_i-cycles in S_{k_i}),

(5) Let $f_\gamma(u)a_\gamma$, $f_\gamma(u) \neq 0$, be a term of $\eta_{\alpha,s}$ with $l(\gamma) = l_\alpha$. Then $\gamma \in C_\alpha \cap P_{\alpha,s}$ (consequently, $f_\gamma(u) = 1$ by part (2) of this lemma).

Proof. (1) This follows from Corollary (3.15), Lemma (3.16), and Remark (2) following Definition (3.21).

(2) This follows from Lemma (3.16), Lemma (3.18), and Remark (2) following Definition (3.21).

(3) This follows from Lemma (3.2) and Remark (2) following Definition (3.21).

(4) This follows from Lemma (3.19) and Lemma (3.22).

(5) This follows from Lemma (3.17) and Remark (2) following Definition (3.21). □

Remarks. (1) We may restate part (4) of Lemma (3.23) as follows: Let C_α be the conjugacy class in S_n corresponding to α. Then

$$\eta_{\alpha,s}|_{u=1} = \text{ the sum of the elements in } C_\alpha \cap P_{\alpha,s}.$$

(2) If we write $\eta_{\alpha,s} = \sum_{w \in P_{\alpha,s}} f_w(u)a_w$ with $f_w(u) \in \mathbf{Q}[u, u^{-1}]$, part (4) implies $f_w(1) = 0$ or 1.

Definition (3.24). Let $\alpha, P_{\alpha,s}$, and $\eta_{\alpha,s}$ be as in Definition (3.21). Define

$$b_{\alpha,s} = N_{S_n, P_{\alpha,s}}(\eta_{\alpha,s}).$$

Also,

$$b_\alpha = N_{S_n, P_\alpha}(\eta_\alpha).$$

Lemma (3.25). *The element $b_{\alpha,s}$ contains only terms of length l_α and longer.*

Proof. In taking the norm of $\eta_{\alpha,s}$, lengths for right multiplications add by Proposition (1.1.2). The left multiplictions can only reduce the length by, at most, the same length as it had been increased by the right multiplications. Thus the result follows from Corollary (3.15). □

Lemma (3.26). *The element $b_{\alpha,s}$ is in $Z(A)$.*

Proof. This is immediate from Lemma (3.23), part (3) and Proposition (2.13). □

Lemma (3.27). *Let $f_\gamma(u)a_\gamma$ be a term of $b_{\alpha,s}$ with $f_\gamma(u) \neq 0$. Suppose $l(\gamma) = l_\alpha$. Then $\gamma \in C_\alpha$.*

Proof. In view of Lemma (3.23), part (5), we can use the same argument on a term a_γ of $\eta_{\alpha,s}$ with $l(\gamma) = l_\alpha$ in taking the norm of $\eta_{\alpha,s}$ to get $b_{\alpha,s}$, as was used in Lemma (3.17), to get this result. □

Corollary (3.28). *Let $f_\gamma(u)a_\gamma$ be a term of $b_{\alpha,s}$ with $l(\gamma) = l_\alpha$. Then $f_\gamma(u)$ is a positive integer.*

Proof. From Lemma (3.27) and the technique in its proof, $\gamma \in C_\alpha$ and a_γ must be realized as $a_{r^{-1}\tau r}$ where r is a distinguished right coset representative

for $P_{\alpha,s}$ in S_n, $\tau \in C_\alpha \cap P_{\alpha,s}$ with $l(\tau) = l_\alpha$ and a_τ is a term of $\eta_{\alpha,s}$ having coefficient one. Thus, in taking the norm we get the product

$$u^{-l(r)}a_{r-1}a_\tau a_r = a_\gamma + \text{ other terms of longer length}$$

and the corollary follows. $\square$

Proposition (3.29). *Let* C_α *be the conjugacy class of* S_n *corresponding to* α *and let* $\underline{C}_\alpha$ *denote the sum of the elements in* C_α. *Then*

$$b_{\alpha,s}|_{u=1} = [N_{S_n}(P_{\alpha,s}): P_{\alpha,s}]\underline{C}_\alpha.$$

Proof. Let $f_\gamma(u)a_\gamma$ be a term of $\eta_{\alpha,s}$ with $f_\gamma(1) \neq 0$ so that $\gamma \in C_\alpha \cap P_{\alpha,s}$ by Remark (1) following Lemma (3.23). Let T be the set of distinguished right coset representatives for $P_{\alpha,s}$ in S_n. Then for $r \in T$, we have

$$a_{r-1}f_\gamma(u)a_\gamma a_r = f_\gamma(u)a_{r-1\gamma r} + f_\gamma(u)(u-1)(\text{other terms})$$

so that only terms like $a_{r-1\gamma r}$ will survive in the specialization of $b_{\alpha,s}$; that is, only terms corresponding to conjugates of γ. From Lemma (3.26) we conclude that $b_{\alpha,s}|_{u=1}$ is some multiple of $\underline{C}_\alpha$. To determine the exact multiple, we will determine the coefficient on a_γ in $b_{\alpha,s}|_{u=1}$. If $r \in N_{S_n}(P_{\alpha,s}) \cap T$, then by Lemma (3.22), part (2), and Lemma (3.23), part (4), we have that $a_{r-1}\eta_{\alpha,s}a_r|_{u=1}$ contains exactly one a_γ. On the other hand, if for $\lambda \in C_\alpha \cap P_{\alpha,s}$ and $r \in T$ we have $r^{-1}\lambda r = \gamma$, then $w^{-1}r^{-1}\lambda rw = w^{-1}\gamma\omega = \lambda$ for some $w \in P_{\alpha,s}$. Thus $rw \in C_{S_n}(\lambda)$. Rewrite $P_{\alpha,s}$ as $S_{m_1}^{v_1} \times S_{m_2}^{v_2} \times \cdots \times S_{m_z}^{v_z}$ where $S_{m_i} \neq S_{m_j}$ for $i \neq j$. Then write λ as $\prod_{j=1}^{z}\prod_{i=1}^{v_j}\lambda_{i,j}$ where $\lambda_{i,j}$ is an m_j-cycle of length $m_j - 1$ inside one of the S_{m_j} factors of $P_{\alpha,s}$. Thus

$$C_{S_n}(\lambda) = S_t \times \prod_{j=1}^{z}\prod_{i=1}^{v_j}\langle\lambda_{i,j}\rangle \times S_{v_1} \times S_{v_2} \times \cdots \times S_{v_z}$$

$$\leq S_t \times P_{\alpha,s} \times S_{v_1} \times S_{v_2} \times \cdots \times S_{v_z}$$

$$= N_{S_n}(P_{\alpha,s}), \quad \text{where } t = n - \sum_{i=1}^{z}v_i m_i.$$

Therefore, $rw \in N_{S_n}(P_{\alpha,s})$ and since obviously $w \in N_{S_n}(P_{\alpha,s})$, it follows that $r \in N_{S_n}(P_{\alpha,s})$. Hence, the coefficient on a_γ in $b_{\alpha,s}|_{u=1}$ is $|N_{S_n}(P_{\alpha,s}) \cap T| = [N_{S_n}(P_{\alpha,s}): P_{\alpha,s}]$ and the proposition is proved. $\square$

Corollary (3.30). *The positive integer coefficient that occurs in Corollary* (3.28) *is* $[N_{S_n}(P_{\alpha,s}): P_{\alpha,s}]$.

Proof. This is immediate from Corollary (3.28) and Proposition (3.29). $\square$

Remark. Since $[N_{S_n}(P_{\alpha,s}) : P_{\alpha,s}] = [N_{S_n}(P_\alpha) : P_\alpha]$, the positive integer coefficient in Corollary (3.28) is independent of s.

Definition (3.31). Define B to be the set $\{b_\alpha \mid \alpha \vdash n\}$, and $B|_{u=1}$ to be the set $\{b_\alpha|_{u=1} \mid \alpha \vdash n\}$.

Corollary (3.32). *The set $B|_{u=1}$ is a basis for $Z(\mathbf{Q}S_n)$ over $\mathbf{Q}$.*

Proof. Immediate from Proposition (3.29). □

Remark. Lemma (3.25) and Lemma (3.27) imply that the elements of B can be put into a *triangular* form with respect to the lengths of the terms involved in the elements of B. This fact is crucial to the proof of the following main theorem.

Theorem (3.33). *The set B is a basis for $Z(A(S_n))$ over $\mathbf{Q}[u, u^{-1}]$.*

Proof. Suppose there is a relation $\sum_{\alpha \vdash n} f_\alpha(u) b_\alpha = 0$ with $f_\alpha(u) \in \mathbf{Q}[u, u^{-1}]$ and not all $f_\alpha(u)$ equal to zero. If $f_\alpha(1) = 0$ for all $f_\alpha(u) \neq 0$, then $u - 1$ is a factor of each $f_\alpha(u) \neq 0$ and we may cancel it. Thus we may assume that $f_\alpha(1) \neq 0$ for some α. But then

$$0 = \left(\sum_{\alpha \vdash n} f_\alpha(u) b_\alpha\right)\bigg|_{u=1} = \sum_{\alpha \vdash n} f_\alpha(1) b_\alpha|_{u=1}$$

contradicts Corollary (3.32). Hence, B is a linearly independent set over $\mathbf{Q}[u, u^{-1}]$. Now, for each partition $\alpha = (k_1, k_2, \ldots, k_t)$ of n, pick an element $\beta_{\alpha,\tau} = a_\tau \in \{a_w \mid w \in S_n\}$ such that $\tau \in C_\alpha$, $l(\tau) = l_\alpha$ and τ is supported on $\{1, 2, \ldots, \sum_{i=1}^{v} k_i\}$ where $k_i = 1$ for $v < i \leq t$. In other words, τ has minimal length in its conjugacy class in S_n and is *left-justified*. Let $S = \{\beta_{\alpha,\tau} \mid \alpha \vdash n\}$. Denote the span of S in $A(S_n)$ over $\mathbf{Q}[u, u^{-1}]$ by $\langle S \rangle$. Let π be the projection of $Z(A(S_n))$ onto $\langle S \rangle$. Considering dimensions over $\mathbf{Q}(u)$, we deduce from the fact mentioned in the remark following Theorem (1.2.5) that the kernel of π is trivial. Thus

$$\langle S \rangle / \pi(\langle B \rangle) \simeq Z(A(S_n)) / \langle B \rangle.$$

From Lemma (3.25) and Lemma (3.27) we have $\pi(B) = S$ so that $\pi(\langle B \rangle) = \langle S \rangle$. Hence, $\langle B \rangle = Z(A(S_n))$ and the theorem is proved. □

Examples. (1) $n = 3$.

$$\eta = N_{S_2, S_1}(a_{(12)(23)}) = a_{(123)} + a_{(132)} + u^{-1}(u-1)a_{(13)}.$$

Let $\beta = a_{(12)} + a_{(23)} + u^{-1}a_{(13)}$. Then

$$B = \{N_{S_3, S_1}(a_1), N_{S_3, S_2}(a_{(12)}), N_{S_3, S_3}(\eta) = \eta\},$$

where

$$N_{S_3, S_1}(a_1) = 6a_1 + 3u^{-1}(u-1)\beta + u^{-2}(u-1)^2\eta$$

and

$$N_{S_3, S_2}(a_{(12)}) = \beta + u^{-1}(u-1)\eta.$$

(2) $n = 4$.

$$\eta = N_{S_3,S_1}(a_{(12)(23)(34)})$$

$$= a_{(1234)} + a_{(1432)} + a_{(1243)} + a_{(1342)}$$

$$+ u^{-2}(u^2 - u + 1)(a_{(1423)} + a_{(1324)})$$

$$+ u^{-1}(u - 1)(a_{(13)(24)} + a_{(124)} + a_{(142)} + a_{(134)} + a_{(143)})$$

$$+ u^{-2}(u - 1)^2 a_{(14)} + u^{-3}(u - 1)(u^2 + 1)a_{(14)(23)}.$$

Let

$$\beta = a_{(12)(34)} + u^{-1}a_{(13)(24)} + u^{-2}a_{(14)(23)},$$

$$\gamma = a_{(123)} + a_{(132)} + a_{(234)} + a_{(243)}$$

$$+ u^{-1}(a_{(124)} + a_{(142)} + a_{(134)} + a_{(143)})$$

$$+ u^{-2}(u - 1)(a_{(1324)} + a_{(1423)}) + 2u^{-2}(u - 1)a_{(14)}$$

$$+ u^{-3}(u - 1)^2 a_{(14)(23)} + u^{-1}(u - 1)(a_{(13)} + a_{(24)})$$

and

$$\tau = a_{(12)} + a_{(23)} + a_{(34)} + u^{-1}(a_{(13)} + a_{(24)}) + u^{-2}a_{(14)}.$$

Then

$$B = \{N_{S_4,S_1}(a_1), N_{S_4,S_2}(a_{(12)}), N_{S_4,S_2 \times S_2}(a_{(12)(34)}), N_{S_4,S_3}(N_{S_2,S_1}(a_{(12)(23)})), \eta\},$$

where

$$N_{S_4,S_1}(a_1) = 24a_1 + 12u^{-1}(u - 1)\tau + 4u^{-2}(u - 1)^2\gamma$$

$$+ 6u^{-2}(u - 1)^2\beta + u^{-3}(u - 1)^3\eta,$$

$$N_{S_4,S_2}(a_{(12)}) = 2\tau + 2u^{-1}(u - 1)\gamma + 2u^{-1}(u - 1)\beta + u^{-2}(u - 1)^2\eta,$$

$$N_{S_4,S_2 \times S_2}(a_{(12)(34)}) = 2\beta + u^{-1}(u - 1)\eta,$$

$$N_{S_4,S_3}(N_{S_2,S_1}(a_{(12)(23)})) = N_{S_4,S_3}(a_{(123)} + a_{(132)} + u^{-1}(u - 1)a_{(13)})$$

$$= \gamma + u^{-1}(u - 1)\eta.$$

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